Rossmoyne SHS Mathematics Department	MATHEMATICS SPECIALIST 3CD	Semester 1 2011 EXAMINATION

NAME:	SOLUTIONS
TEACHER:	Mrs Benko Mr Birrell Ms Robinson

Section Two: Calculator-assumed

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for this section: 100 minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

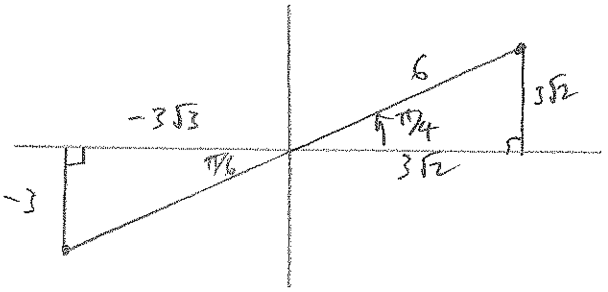
Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

10. (2, 3 = 5 marks)

Given $z = 3\sqrt{2} + 3\sqrt{2}\,i$ and $w = -\sqrt{27} - 3i$.

(a) Express z and w in exact polar form.



$z = 6 \text{ cis}\left(\frac{\pi}{4}\right)$ ✓

$w = 6 \text{ cis}\left(-\frac{5\pi}{6}\right)$ ✓

R/W

P/t from (a)

(b) If v = a cis b where a and b are real constants, find a and b given that $vz = 42 \text{ cis } \dfrac{\pi}{20}$

$a \text{ cis } b \times 6 \text{ cis } \pi/4 = 42 \text{ cis } \pi/20$

$a \text{ cis } b = 7 \text{ cis}\left(\pi/20 - \pi/4\right)$

$a \text{ cis } b = 7 \text{ cis}\left(-\frac{4\pi}{20}\right)$ ✓

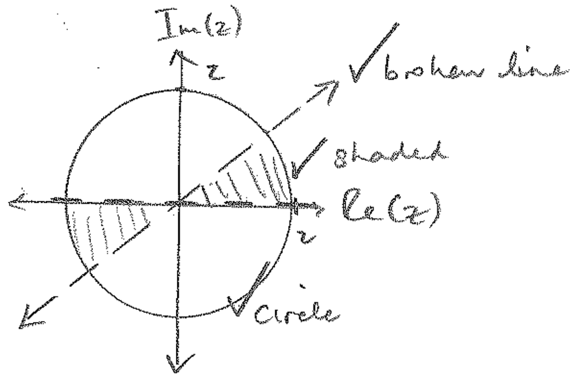
$\therefore\ a = 7$ ✓ and $b = -\pi/5$ ✓

$\boxed{5}$

11. (3 marks)

Sketch the locus of all points on the complex plane which satisfy:

$$\frac{\text{Re}(z)}{\text{Im}(z)} > 1 \text{ and } |z| \le 2.$$



$\frac{x}{y} > 1$

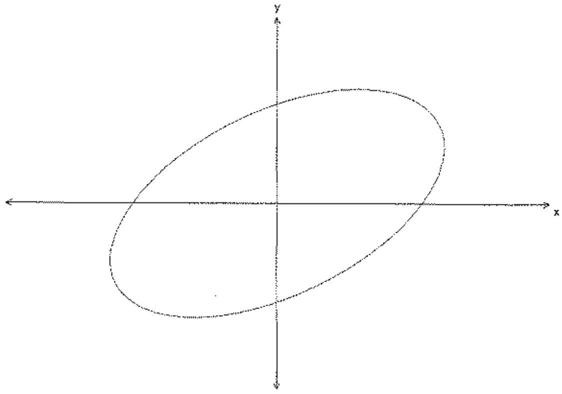
$y < x$ for $x, y > 0$

$y > x$ for $x, y < 0$.

$\boxed{3}$

12. (5 marks)

The elliptical graph shown below has the equation $x^2 - xy + y^2 = 9$.



Use implicit differentiation to determine the points on the graph where the tangent is vertical.

$$x^2 - xy + y^2 = 9 \quad \text{— (1)}$$

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(9)$$

$$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0 \quad ✓ \qquad \boxed{5}$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x} \quad ✓$$

If tangent vertical then $\frac{dy}{dx}$ undefined

i.e. $2y - x = 0$ ✓

$x = 2y$ subst. into — (1)

$$(2y)^2 - 2y \cdot y + y^2 = 9$$
$$y^2 = 3$$
$$y = \pm\sqrt{3} \quad ✓$$

∴ Coordinates : $(2\sqrt{3}, \sqrt{3})$, $(-2\sqrt{3}, -\sqrt{3})$ ✓

Must have both

13. (4 marks)
The vertices of the triangle ABC have coordinates $A(-3, 1, 10)$, $B(7, 1, 0)$ and $C(-7, 5, 2)$.
The point D divides $\overrightarrow{AC}$ internally in the ratio 5:3. Find the vector $\overrightarrow{BD}$.

$\overrightarrow{AC} = \begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix}$ ✓

$\overrightarrow{AD} = \frac{5}{8}\overrightarrow{AC} = \begin{pmatrix} -5/2 \\ 5/2 \\ -5 \end{pmatrix}$ ✓

$\overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA} = \begin{pmatrix} -11/2 \\ 7/2 \\ 5 \end{pmatrix}$ ✓

$\therefore \overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} -25/2 \\ 5/2 \\ 5 \end{pmatrix}$ ✓

or $\overrightarrow{BD} = \overrightarrow{BA} + \frac{5}{8}\overrightarrow{AC} = \begin{pmatrix} -10 \\ 0 \\ 10 \end{pmatrix}$ ✓ $+ \frac{5}{8}$ ✓ $\begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix}$ ✓ $= \begin{pmatrix} -25/2 \\ 5/2 \\ 5 \end{pmatrix}$ ✓

$\boxed{4}$

14. (5 marks)
Use proof by exhaustion to prove that:
For integer x, x > 1, $x^3 - x$ is always a multiple of 6.

For mult of 6 must be of form
$6n + c$ for $c = 0, 1, 2, 3, 4, 5$. ✓

$\boxed{5}$

$(6n+c)^3 - (6n+c)$ ✓

$= (6n)^3 + 3(6n)^2(c) + 3(6n)(c)^2 + c^3 - 6n - c$

$= \underbrace{6\left[6^2n^3 + 3(6n^2)(c) + 3nc^2 - n\right]}_{\text{mult. of } 6} + c^3 - c$ ✓

Testing for c.
For $c = 0$, $c^3 - c = 0$
$c = 1$, $c^3 - c = 0$
$c = 2$, $c^3 - c = 8 - 2 = 6 = 1 \times \underline{6}$
$c = 3$, $c^3 - c = 24 = 4 \times \underline{6}$
$c = 4$, $c^3 - c = 60 = 10 \times \underline{6}$
$c = 5$, $c^3 - c = 120 = 20 \times \underline{6}$

$\therefore$ Mult. of 6 ✓ ✓

should be $3^x + 17$.

15. (4 marks)
Show how to solve $3^{x+1} = 3^x - 17$ exactly using natural logarithms.

$$3 \cdot 3^x - 3^x = 17$$
$$2 \cdot 3^x = 17$$
$$3^x = 17/2$$
$$x \ln 3 = \ln(17/2)$$
$$x = \frac{\ln(17/2)}{\ln 3}$$

[NB Pay no sol'n if appropriate working ---.]

$\boxed{4}$

16. (2, 2, 2, 4, 3 = 13 marks)
Find the following indefinite integrals using calculus techniques:

(a) $\int \frac{30x}{3x^2-5}dx = 5 \ln|3x^2-5| + c$

Ans only OK

(c) $\int \frac{10}{\sqrt{5x-3}}dx = \int 10(5x-3)^{-1/2} dx$

$= 4(5x-3)^{1/2} + c$

$= 4\sqrt{5x-3} + c$

(d) $\int 16\cos(2x)e^{\sin(2x)}(3+e^{\sin(2x)})^3 dx$

$= \frac{8(3+e^{\sin 2x})^4}{4} + c$

$= 2(3+e^{\sin 2x})^4 + c$

(e) $\int 2\sin^3(1-x)dx$

$= \int 2\sin^2(1-x) \cdot \sin(1-x)\, dx$

$= \int (2 - 2\cos^2(1-x)) \cdot \sin(1-x)\, dx$

$= \int 2\sin(1-x) - 2\cos^2(1-x) \cdot \sin(1-x)\, dx$

$= 2\cos(1-x) - \frac{2}{3}\cos^3(1-x) + c$

Ans. only 1 mark max

(f) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$ (Hint: Let u = 1 + $\sqrt{x}$)

$u = 1 + x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$2\sqrt{x}\, du = dx$

$= \int \frac{1}{\cancel{\sqrt{x}}(u)} \cdot 2\cancel{\sqrt{x}}\, du$

$= 2\int \frac{1}{u}\, du$

$= 2\ln|u| + c$

$= 2\ln|1+\sqrt{x}| + c$

$\boxed{13}$

−1 overall if missing +c on more than one occasion

17. (4 marks) $\cos 2t = \sqrt[3]{x}$

Find $\frac{dy}{dx}$, in terms of x, given $x = \cos^3 2t$ and $y = 4\sin^2 2t$.

Show sufficient working to justify your answer.

$$\frac{dx}{dt} = 3\cos^2 2t\,(-\sin 2t) \cdot 2$$
$$= -6\sin 2t \cdot \cos^2 2t \checkmark$$

$$\frac{dy}{dt} = 8\sin 2t \cdot \cos 2t \cdot 2$$
$$= 16\sin 2t \cdot \cos 2t \checkmark$$

$$\therefore \frac{dy}{dx} = \frac{16\sin 2t\cos 2t}{-6\sin 2t\cos^2 2t} \checkmark$$
$$= \frac{8}{-3\cos 2t}$$
$$= -\frac{8}{3\sqrt[3]{x}} \checkmark$$

$\boxed{4}$

18. (4 marks)

Solve $10x^2 y \frac{dy}{dx} = 7$ given $y = 2$ when $x = 1$.

$$y\,dy = \frac{7}{10x^2}\,dx$$

$$\int y\,dy = \frac{7}{10}\int \frac{1}{x^2}\,dx \checkmark$$

$$\frac{y^2}{2} = \frac{7}{10}\cdot\left(-\frac{1}{x}\right) + c \checkmark$$

$$(1,2) \Rightarrow \frac{4}{2} = \frac{7}{10}\cdot(-1) + c$$

$$2 + \frac{7}{10} = c$$

$$\frac{27}{10} = c \checkmark$$

$$\therefore \frac{y^2}{2} = -\frac{7}{10x} + \frac{27}{10} \checkmark$$

$\boxed{4}$

$$\left(5y^2 = -\frac{7}{x} + 27\right)$$

19. (5 marks)

The solution to the differential equation $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = -2x + 5$ is given as

$y = e^{\frac{x}{2}} + e^x + ax + b$.

Find the values of a and b. Show your reasoning clearly.

$\frac{dy}{dx} = \frac{1}{2}e^{\frac{x}{2}} + e^x + a$ ✓

$\frac{d^2y}{dx^2} = \frac{1}{4}e^{\frac{x}{2}} + e^x$ ✓

$\therefore 2(\frac{1}{4}e^{\frac{x}{2}} + e^x) - 3(\frac{1}{2}e^{\frac{x}{2}} + e^x + a) + (e^{\frac{x}{2}} + e^x + ax + b) = -2x + 5$

$-3a + ax + b = -2x + 5$ ✓

$(a)x + (b - 3a) = (-2)x + (5)$

$\therefore \underline{a = -2}$ and $b - 3(-2) = 5$

$\underline{b = -1}$ ✓ ✓

$\boxed{5}$

20. (5 marks)

Show that $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{2+\sin\theta}\,d\theta = \ln\frac{3}{2}$.

$LHS = \int_0^{\pi/2} \frac{\cos\theta}{2+\sin\theta}\,d\theta$

$= \left[\ln|2+\sin\theta|\right]_0^{\pi/2}$ ✓✓

$= \ln(2+\sin\pi/2) - \ln(2+\sin 0)$ ✓

$= \ln(3) - \ln(2)$ ✓

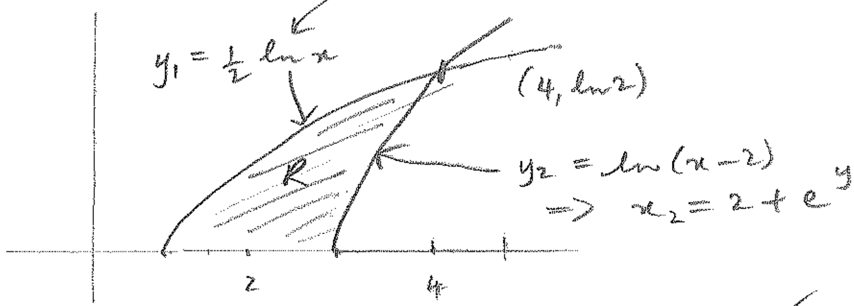
$= \ln\left(\frac{3}{2}\right)$ ✓

$= RHS$

$\boxed{5}$

21. (4, 3, 1 = 8 marks)
The x axis and the curves $y = \ln(x\text{-}2)$ and $2y = \ln x$ fully enclose a region R in the first quadrant.

(a) By first considering areas between the curves and the x axis, express the area of R in terms of integrals with respect to x.



$$\text{Area} = \int_1^4 \left(\tfrac{1}{2}\ln x\right) dx - \int_3^4 \ln(x-2)\, dx$$

✓✓ limits

(b) By considering areas between the curves and the y axis, express the area of R terms of an integral with respect to y.

$$\text{Area} = \int_0^{\ln 2} (x_2 - x_1)\, dy$$

$$= \int_0^{\ln 2} \left(2 + e^{y} - e^{2y}\right) dy$$

8

(c) Evaluate the area of R giving your answer correct to three decimal places.

Area = 0.886 (to 3dp's) ✓

22. (6, 3, 6 = 15 marks)

Consider the plane Π_1: $\mathbf{r}.\begin{pmatrix}6\\2\\-3\end{pmatrix}=4$

(a) Determine the equation of the plane(s) parallel to Π_1 and exactly 14 units away.

Point on Π_1 : $\begin{pmatrix}0\\2\\0\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=4$ {May use $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ or many others}

$\left|\begin{pmatrix}6\\2\\-3\end{pmatrix}\right|=7 \Rightarrow$ require $2\begin{pmatrix}6\\2\\-3\end{pmatrix}$ for 14 units.

Two Planes are possible

Point on plane: $\begin{pmatrix}0\\2\\0\end{pmatrix}+2\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}12\\6\\-6\end{pmatrix}$ $\qquad$ $\begin{pmatrix}0\\2\\0\end{pmatrix}-2\begin{pmatrix}6\\2\\-3\end{pmatrix}=\begin{pmatrix}-12\\-2\\6\end{pmatrix}$

$\underset{\sim}{r}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=\underset{\sim}{a}$ $\qquad$ $\underset{\sim}{r}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=\underset{\sim}{b}$

$\begin{pmatrix}12\\6\\-6\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=102$ $\qquad$ $\begin{pmatrix}-12\\-2\\6\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=-94$

$\therefore \underset{\sim}{r}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=102$ $\qquad$ $\therefore \underset{\sim}{r}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=-94$

The line L_1: $\mathbf{r}=\begin{pmatrix}-2\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\4\end{pmatrix}$ intersects Π_1 at the point P.

(b) Determine the coordinates of the point P.

$\begin{pmatrix}-2-\lambda\\1+2\lambda\\4\lambda\end{pmatrix}\cdot\begin{pmatrix}6\\2\\-3\end{pmatrix}=4$

$-12-6\lambda+2+4\lambda-12\lambda=4$

$-10-14\lambda=4$

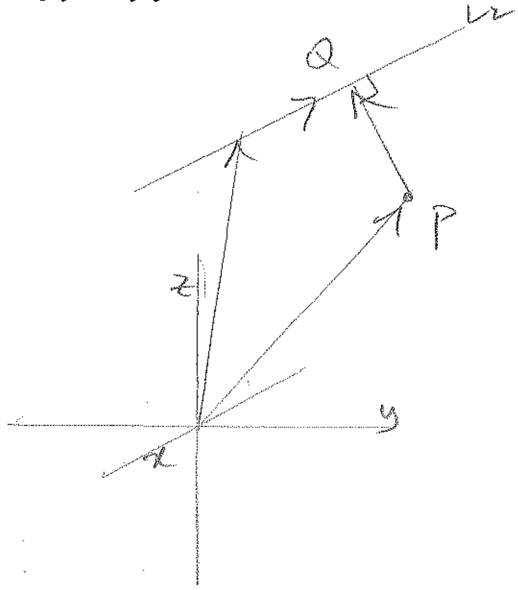
$-14\lambda=14$

$\lambda=-1$

$\therefore$ Coordinates are: $x=-1$, $y=-1$, $z=-4$ or $\begin{pmatrix}-1\\-1\\-4\end{pmatrix}$

(c) Determine the exact minimum distance between the line

$$L_2: \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$ and the point P. You must **fully** justify your solution.



Let closest point on L2 be Q.

$$\overrightarrow{PQ} = -\begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} + \begin{pmatrix} 2+\mu \\ 1+3\mu \\ -3+5\mu \end{pmatrix}$$

$$= \begin{pmatrix} 3+\mu \\ 2+3\mu \\ 1+5\mu \end{pmatrix}$$ ✓✓

$$\begin{pmatrix} 3+\mu \\ 2+3\mu \\ 1+5\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = 0$$ ✓

$$3+\mu + 6+9\mu + 5+25\mu = 0$$

$$14 + 35\mu = 0$$

$$\mu = -\frac{14}{35}$$

$$\mu = -2/5$$ ✓

$$\overrightarrow{PQ} = \begin{pmatrix} 2.6 \\ 0.8 \\ -1 \end{pmatrix}$$ ✓

$\boxed{15}$

$$|\overrightarrow{PQ}| = \sqrt{\frac{45}{5}} \text{ units.}$$ ✓

$$\approx 2.898 \text{ units}$$

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